# V. THE GRAVITATIONAL LAGRANGIAN

## 1. Introduction

So far (Chapters III and IV) my discussion of metric-connection theories has been limited to the kinematics of the gravitational fields and the kinematics and dynamics of the matter. Some of this discussion depended on the choice of matter Lagrangian and minimal coupling, but all of it was independent of the choice of gravitational Lagrangian.

In this chapter, I discuss the choice of gravitational Lagrangian. Specifically, I list experimental and theoretical criteria which ought to be satisfied by a good theory of gravity. Some of these are easily satisfied by choosing the Lagrangian as a scalar function of certain variables; for example

(i) the field equations should involve no higher than second derivatives of the gravitational field variables; and

(ii) the field equations and certain differential identities on the gravitational variables should guarantee that the matter variables automatically satisfy the Noether conservation laws.

Other criteria are straightforward but tedius to check; such as

(iii) the existence of a good initial value formulation; and

(iv) agreement with Newtonian and post-Newtonian experiments.

Still other criteria require a proof of the existence or uniqueness of certain solutions to the field equations; including

(v) the existence of a Birkhoff theorem;

(vi) the existence of cosmological solutions; and

(vii) the existence of a unique gravitational ground state.

At the other extreme, one criteria,

(viii) the existence of a good (unitary and renormalizable) quantum theory,

is very difficult to satisfy and has not yet been satisfied by any proposed theory of gravity.

In addition to the theoretical and experimental criteria, I make several aesthetic and simplicity assumptions. For simplicity (and to allow for the existence of spinors) I restrict to metric-Cartan connection theories. Skinner and Gregorash [1976] and Aldersley [1977a] have investigated such theories under the assumption that the gravitational variables are the metric and torsion. Unlike their work, I use the gauge theory analogy of Chapter II to justify the assumption:

(ix) The gravitational field variables are the components of the orthonormal frame and the mixed components of the Cartan connection.

In conjunction with the assumption of second order field equations, this leads to the restriction of the gravitational Lagrangian to the sum of the Christoffel scalar curvature, $\tilde{R}$, and an arbitrary scalar function, $\bar{L}(g_{\alpha\beta}, \hat{R}^{\alpha}{}_{\beta\gamma\delta}, Q^{\alpha}{}_{\gamma\delta})$, of the metric, the Cartan curvature, and the torsion tensors. This is a very different class of theories than those obtained by Skinner, Gregorash and Aldersley. For simplicity, I assume:

(x) The function, $\bar{L}$, is a quadratic polynomial in the Cartan curvature and torsion tensors.

This restricts the gravitational Lagrangian to the twelve parameter family (derived in Section V.3d):

$$L_G = -\frac{\hbar c}{8\pi L^2}\Lambda - s\,\frac{\hbar c}{16\pi L^2}(c_1\,\hat{R} + c_2\,\tilde{R})$$

$$+ \frac{\hbar c}{16\pi}(\ a_1\,\hat{R}\,\hat{R} + a_2\,\hat{R}_{\beta\delta}\,\hat{R}^{\beta\delta} + a_3\,\hat{R}_{\alpha\beta\gamma\delta}\,\hat{R}^{\alpha\beta\gamma\delta}$$

$$+ a_4\,\hat{R}_{\beta\delta}\,\hat{R}^{\delta\beta} + a_5\,\hat{R}_{\alpha\beta\gamma\delta}\,\hat{R}^{\gamma\delta\alpha\beta} + a_6\,\hat{R}_{\alpha\beta\gamma\delta}\,\hat{R}^{\alpha\gamma\beta\delta})$$

$$- s\,\frac{\hbar c}{16\pi L^2}(b_1\,Q^{\alpha}{}_{\alpha\delta}\,Q^{\gamma}{}_{\gamma}{}^{\delta} + b_2\,Q^{\alpha}{}_{\gamma\delta}\,Q_{\alpha}{}^{\gamma\delta} + b_3\,Q^{\alpha}{}_{\gamma\delta}\,Q^{\gamma}{}_{\alpha}{}^{\delta}). \quad (1)$$

It is not known whether any of the theories in this twelve parameter family satisfy all of the experimental and theoretical criteria listed above (especially the quantizability condition). I consider it worthwhile to study all of these theories at both the classical and quantum levels. As far as the classical field equations are concerned, one of the parameters, $a_1$, $a_4$, or $a_5$, is arbitrary because of the "Euler-Gauss-Bonnet identity:"

$$\delta \int P\,\sqrt{-\bar{g}}\,d^4x = 0, \quad (2)$$

where

$$P = \epsilon_{\alpha\beta\kappa\lambda}\,\epsilon^{\gamma\delta\mu\nu}\,\hat{R}^{\alpha\beta}{}_{\gamma\delta}\,\hat{R}^{\kappa\lambda}{}_{\mu\nu}\ .$$

$$= -4\ (\,\hat{R}\,\hat{R} - 4\,\hat{R}_{\beta\delta}\,\hat{R}^{\delta\beta} + \hat{R}_{\alpha\beta\gamma\delta}\,\hat{R}^{\gamma\delta\alpha\beta}\,)\ . \quad (3)$$

All of the theories in the family (1) have automatic Noether conservation laws. Further, they all have field equations involving no higher than second derivatives of the frame and connection. In fact, if $c_2 = b_1 = b_2 = b_3 = 0$, then the Lagrangian does not depend on any derivatives of the frame, and so the frame acts as a Lagrange multiplier. The quantum propagators, for the theory with $\Lambda = c_2 = b_1 = b_2 = b_3 = 0$, have been investigated by Neville [1978].

One parameter, $\Lambda$, may be identified as a cosmological constant. The Newtonian limit should determine another parameter, probably some combination of the $b_i$'s and $c_i$'s. In the post-Newtonian limit, it may be possible to identify other parameters with the parameters of the parametrized post-Newtonian formalism. (See Will [1974].)

The analogy with the Yang-Mills Lagrangian is ambiguous and has been used to justify any of the terms:

$$\hat{R}^{\alpha}{}_{\beta\gamma\delta}\,\hat{R}^{\beta}{}_{\alpha}{}^{\gamma\delta}, \quad \tilde{R}, \quad \hat{R}, \quad Q^{\alpha}{}_{\gamma\delta}\,Q_{\alpha}{}^{\gamma\delta}, \quad (Q^{\alpha}{}_{\gamma\delta}\,Q_{\alpha}{}^{\gamma\delta} - 2\,Q^{\alpha}{}_{\alpha\delta}\,Q^{\gamma}{}_{\gamma}{}^{\delta}). \tag{4}$$

Cho [1976a] argues that $\tilde{R}$ is the Yang-Mills Lagrangian for the translation group. On the other hand, Yang [1974] regards $\hat{R}^{\alpha}{}_{\beta\gamma\delta}\,\hat{R}^{\beta}{}_{\alpha}{}^{\gamma\delta}$ as the Yang-Mills Lagrangian for the Lorentz group. Combining these, one obtains the Lagrangian,

$$L_G = -\,s\,\frac{\hbar c}{16\pi L^2}\,\tilde{R} - \frac{\hbar c}{16\pi\alpha_G}\,\hat{R}^{\alpha}{}_{\beta\gamma\delta}\,\hat{R}^{\beta}{}_{\alpha}{}^{\gamma\delta}, \tag{5}$$

as a Yang-Mills Lagrangian for the Poincare group. However, in Section 3c, I rule out this theory because it implies separate conservation of spin and orbital angular momentum. The problem may be that Lagrangian (5) ignores the coupling between the translation and Lorentz groups. In Chapter VI, I study the theory with the Lagrangian,

$$L_G = -\,s\,\frac{\hbar c}{16\pi L^2}\,\hat{R} - \frac{\hbar c}{16\pi\alpha_G}\,\hat{R}^{\alpha}{}_{\beta\gamma\delta}\,\hat{R}^{\beta}{}_{\alpha}{}^{\gamma\delta}, \tag{6}$$

and show that this theory has a Birkhoff theorem, has at least one cosmological solution, and has Minkowski space as the unique gravitational ground state. The same theory has been proposed and investigated independently by Mansouri and Chang [1976] and by Fairchild [1977].

In Section II.5, I argued that the Lagrangian,

$$L_G = - s \frac{\hbar c}{16\pi L^2} b_2 Q^{\alpha}{}_{\gamma\delta} Q_{\alpha}{}^{\gamma\delta} - \frac{\hbar c}{16\pi\alpha_G} \hat{R}^{\alpha}{}_{\beta\gamma\delta} \hat{R}^{\beta}{}_{\alpha}{}^{\gamma\delta}, \tag{7}$$

is even more appropriate to the Poincare group than Lagrangian (6). Similarly, Hehl, Ne'eman, Nitsch and von der Heyde [1978] argue for the Lagrangian,

$$L_G = s \frac{\hbar c}{16\pi L^2} (Q^{\alpha}{}_{\gamma\delta} Q_{\alpha}{}^{\gamma\delta} - 2 Q^{\alpha}{}_{\alpha\delta} Q^{\gamma}{}_{\gamma}{}^{\delta}) - \frac{\hbar c}{16\pi\alpha_G} \hat{R}^{\alpha}{}_{\beta\gamma\delta} \hat{R}^{\beta}{}_{\alpha}{}^{\gamma\delta}. \tag{8}$$

I now discuss each of the criteria in more detail.

## 2. Experimental Criteria

### a. Newtonian Limit

The most important requirement for any new theory of gravity is that it must agree with all present day experiments. Most of these occur in the Newtonian limit when (i) the velocities are small, ($c=1$, $G\neq 1$, $s=-1$)

$$v \sim \mathcal{O}(\varepsilon), \tag{1}$$

and (ii) the momentum densities and stresses are small compared to the energy densities,

$$|T_{ok}|/T_{oo} \sim \mathcal{O}(\varepsilon), \tag{2}$$

$$|T_{jk}|/T_{oo} \sim \mathcal{O}(\varepsilon^2). \tag{3}$$

Here $T_{\mu\nu}$ are the orthonormal components of the energy-momentum tensor in a Fermi coordinate system based on an approximate center of mass of the system under consideration. These conditions are satisfied in the solar system with $\varepsilon \sim 10^{-3}$. To avoid ratios, I introduce a typical length scale, R, so that

$$GT_{oo} \sim \mathcal{O}(\varepsilon^2)/R^2, \tag{4}$$

$$GT_{ok} \sim \mathcal{O}(\varepsilon^3)/R^2, \tag{5}$$

$$GT_{jk} \sim \mathcal{O}(\varepsilon^4)/R^2. \tag{6}$$

To say that a theory has a good Newtonian limit means that it is possible to *consistently* assign orders to each of the components of the gravitational field and to identify one gravitational variable as the Newtonian potential, $\Phi \sim \mathcal{O}(\varepsilon^2)$, so that the Newtonian equations of motion,

$$\ddot{x} = - \nabla\Phi , \qquad (7)$$

are satisfied to $\mathcal{O}(\varepsilon^2)/R$ and the Newtonian field equation,

$$\nabla^2\Phi = 4\pi G T_{oo} , \qquad (8)$$

is satisfied to $\mathcal{O}(\varepsilon^2)/R^2$. Equation (7) describes how a massive, spinless and rotationless , point, test body behaves in a gravitational field, while equation (8) describes how the matter produces the gravitational field. Boundary conditions are specified so that far away from the system of interest, or $\varepsilon = 0$, spacetime is empty, the metric is Minkowski and the defect tensor vanishes.

In Cor. IV.1 of Sec. IV.4, I discussed the motion of massive, spinless and rotationless, point, test bodies in the context of metric-Cartan connection theories whose matter satisfies the Noether conservation laws (IV.3.1) and (IV.3.2). I showed that in these theories, these test bodies move on geodesics of the Christoffel connection, just as they do in Einstein's theory. These results are independent of the choice of gravitational Lagrangian or gravitational field equations. Hence for any metric-Cartan connection theory, just as for Einstein's theory, equation (7) is satisfied to $\mathcal{O}(\varepsilon^2)/R$ provided the Newtonian potential is identified as

$$\Phi = - \frac{1}{2} (g_{oo} + 1) + \mathcal{O}(\varepsilon^4), \qquad (9)$$

and the components of the metric are assigned the orders

$$g_{oo} = -1 - 2\Phi + \mathcal{O}(\varepsilon^4), \tag{10}$$

$$g_{ok} = \mathcal{O}(\varepsilon^3), \tag{11}$$

$$g_{jk} = \delta_{jk} + \mathcal{O}(\varepsilon^2). \tag{12}$$

This is independent of the orders assigned to the components of the torsion.

To check that equation (8) is satisfied to $\mathcal{O}(\varepsilon^2)/R$, one expands the gravitational field equations in powers of $\varepsilon$ and verifies that equation (8) appears as some combination of these equations. This requires the gravitational Lagrangian of the particular theory and, as far as I can tell, must be checked separately for each new theory.

One must also check the consistency of the assignment of orders to the gravitational variables. To do this, one assumes the assigned orders, uses the gravitational field equations to compute the gravitational variables in terms of the matter variables appearing as sources, and verifies that the gravitational variables have the assumed orders. As far as I can tell, this must also be checked separately for each new theory.

b. Laboratory and Solar System Experiments

In addition to the Newtonian experiments, a new theory of gravity must also agree with the experiments designed to check general relativity. The analysis of these experiments may be divided into answering two questions about the predictions of the theory:

(i) What gravitational fields are present in these experiments?

(ii) How does matter move in these gravitational fields?

The answer to question (i) depends on the gravitational Lagrangian or gravitational field equations of the particular theory. However, the answer to question (ii) is independent of the choice of gravitational Lagrangian, although it does depend on the choice of gravitational variables and the method of coupling the gravitational field to the matter Lagrangian or matter equations of motion.

Since it is simpler, I discuss question (ii) first. All present experiments measure the motion of either light waves or massive test bodies with no net microscopic spin. In Cor. IV.3 of Sec. IV.4, I discussed the motion of these types of matter in general gravitational fields in the context of metric-Cartan connection theories whose matter satisfies the Noether conservation laws (IV.3.1) and (IV.3.2). For such theories, I showed that in a fixed gravitational field, light waves and massive spinless test bodies move in the same manner as they do in Einstein's theory. Hence, if Einstein's theory and a metric-Cartan connection theory predict the same metric field for a given experiment then that experiment cannot distinguish between the theories.

This brings us back to question (i). For some experiments (e.g. the gravitational red shift of light waves and the uniqueness of free fall for massive bodies) the gravitational field is essentially Newtonian and so is the same for any theory of gravity with a good Newtonian limit.

Other experiments (e.g. the deflection and time delay of light by the sun, the precession of the perihelion of planets, and the precession of a gyroscope) measure the non-Newtonian aspects of the gravitational field. For these experiments the gravitational field must be investigated separately for each new theory. However, there is one shortcut.

For every experiment so far performed, the relevant gravitational field consists of Newtonian and post-Newtonian contributions from the sun and Newtonian contributions from the planets. These experiments are consistent with the statement that the metric in the solar system agrees to post-Newtonian order with a Schwarzschild metric centered at the sun except for non-spherical but Newtonian perturbations due to the planets. Consequently, a new theory with a good Newtonian limit will also agree with these relativistic experiments if

(i) the theory has a Birkhoff theorem, i.e. the theory has a unique spherically symmetric vacuum solution,

(ii) the metric in this spherically symmetric vacuum solution agrees with the Schwarzschild metric to post-Newtonian order, and

(iii) this spherically symmetric vacuum solution is stable under non-spherical perturbations.

Requirement (i) guarantees that there is no ambiguity in the experimental predictions of the theory. Since the measured non-spherical perturbations are Newtonian and the new theory has a good Newtonian limit, requirements

(ii) and (iii) guarantee that the predictions of the theory agree with present experiments.

Thus, experiments using light waves and massive, spinless, test bodies can only be used to distinguish between a theory satisfying these conditions and Einstein's theory if they measure either

(i) the terms in the spherically symmetric metric of higher than post-Newtonian order, (this will not produce a distinction if the unique spherically symmetric vacuum solution of the new theory has precisely the Schwarzschild metric.) or

(ii) the non-Newtonian, non-spherical perturbations to the spherical metric.

For metric-Cartan connection theories, it should be possible to analyze these experiments within the parametrized post-Newtonian formalism since the theoretical predictions only involve the metric. The PPN-parameters for these theories should be calculable from their gravitational field equations.

To go outside of the PPN-formalism and measure torsion effects, one would have to perform experiments using massive bodies with a non-zero net microscopic spin such as a ferromagnet or neutron star. As discussed in Section IV.5 , even these experiments do not seem promising with present technology.

c. Cosmological Observations

In addition to agreeing with the Newtonian and relativistic experiments, a new theory of gravity should be compatible with the cosmological observations that the universe is essentially homogeneous and isotropic on a large scale, is expanding, and has a 3°K black-body spectrum of background microwave radiation. This requirement will be satisfied if the theory admits spatially homogeneous and isotropic solutions for reasonable equations of state which are expanding from a dense hot state.

## 3. Theoretical and Aesthetic Criteria

### a. Initial Value Formulation

Any physical theory should have a good initial value formulation:

(i) It must be possible to divide the equations for the theory into constraint equations satisfied by the field variables at each instant of time, and evolution equations relating the field variables at different times. (An instant of time is simply a 3-dimensional spacelike surface.)

(ii) The evolution equations must guarantee that if the constraint equations are satisfied at an initial time, then they are satisfied at all nearby times.

(iii) The theory must be consistent in that there must exist at least one solution to the constraint equations, or equivalently, at least one solution to the full theory.

Any gauge freedom in the theory shows up as arbitrary gauge functions in the evolution equations.

The existence of a good initial value formulation for a theory implies that predictions can be made from the theory. Specifically, if one knows a certain amount about a system at the present time, namely the initial data, then one can predict what will happen to the system for some time into the future. Each set of initial data, satisfying the constraints, can be evolved into a full solution of the theory which is unique up to the choice of the gauge functions. For sufficiently simple systems, this technique for creating solutions can be used to model the behavior of the system on a computer.

The decomposition into evolution and constraint equations can also be used to count and in principle isolate the dynamical degrees of freedom. Roughly speaking, some field variables can be expressed in terms of the gauge functions and so are arbitrary functions of time. Other field variables can be regarded as the solutions of the constraint equations and so are completely determined in terms of the remaining variables. Qualitatively, these constrained variables describe the Coulombic aspects of the theory. The remaining field variables are the dynamical degrees of freedom which may be freely specified at the initial time and then uniquely evolved in time. Qualitatively, the dynamical degrees of freedom describe the wave properties of the theory.

In a frame-connection theory of gravity one would like to know how many and which of the frame and connection variables are dynamic.

The isolation of the dynamical degrees of freedom is a useful step in the canonical quantization of a theory since it is only these variables which should be quantized. Canonical quantization also requires that the initial value formulation is canonical in that the dynamical degrees of freedom form a phase space and their evolution equations are in the form of Hamilton's equations. The standard way to put a Lagrangian or Hamiltonian theory into a canonical initial value form is to apply the Bergmann-Dirac procedure.

When applied to a Lagrangian theory the Bergmann-Dirac procedure assumes that the Lagrangian contains no higher than first time derivatives of the variables. This can always be achieved by introducing new variables and Lagrange multiplies. Unfortunately, the new variables may not have the physical or geometrical relevance of the original variables. In Einstein's theory, the second time derivatives of the metric are eliminated from the scalar curvature Lagrangian by the addition of a divergence.

### b. Second Order Field Equations

Throughout this section, when I refer to the gravitational field equations, I mean the Euler-Lagrange equations obtained by varying the Lagrangian with respect to the appropriate set of gravitational variables.

It is standard to assume that a theory of gravity should satisfy the condition:

(i) The gravitational field equations involve no higher than second derivatives of the gravitational variables.

Notice that this condition is vacuous unless the gravitational variables are specified since it is always possible to define new variables as the derivatives of the old variables and to derive these definitions from the Lagrangian by introducing Lagrange multipliers.

Many authors (Weyl [1919,21], Cartan [1922, 23, 24, 25], Sciama [1962], Kibble [1961], Hehl [1973,74], Trautman [1972a,b,c,73a], etc.) have investigated metric-connection theories which satisfy the condition:

(ii) The gravitational field equations involve no higher than second derivatives of the metric, $g_{ab}$, and the defect, $\lambda^{a}{}_{bc}$. (The defect may be replaced by the torsion, $Q^{a}{}_{bc}$, and/or the covariant derivative of the metric, $\nabla_a g_{bc}$.)

Skinner and Gregorash [1976] and Aldersley [1977a,b] have found all such theories with a metric-compatible connection.

I find an entirely different class of metric-connection theories by requiring them to satisfy the condition:

(iii) The gravitational field equations involve no higher than second derivatives of the components of the orthonormal frame, $\theta^{\alpha}{}_{a}$, and the mixed components of the connection, $\Gamma^{\alpha}{}_{\beta a}$.

As discussed in Section II.4, I regard these variables as the most analogous to the connection in a gauge theory.

The usual justification for making an assumption such as (i), (ii), or (iii) is that it is necessary to a good initial value formulation. I do not believe such an assumption is necessary. However, I assume (iii) anyway because it makes the mathematics simpler and cuts down on the number of possible gravitational Lagrangians. There are still a very large number of Lagrangians satisfying (iii).

At the end of this subsection I prove two theorems which show that there is a large class of Lagrangians which satisfy (iii) but do not satisfy (ii). These theorems only apply to gravitational theories with a Cartan connection. To understand these theorems, one needs the definition of a strictly local function.

To say that $f = f(g_i)$ is a strictly local function of the functions, $g_i$, means that the value of f at a point, x, in spacetime depends only on the values of the $g_i$'s at the point, x. It is independent of the values of the $g_i$'s at any other points and independent of the derivatives of the $g_i$'s at x.

Theorem V.2 shows that if the gravitational Lagrangian is a strictly local scalar function of the metric, Cartan curvature, and torsion then it satisfies (iii). Notice that the gravitational Lagrangian does not even need to be a polynomial. In contrast, Skinner and Gregorash have shown that such a Lagrangian will not satisfy (ii) unless it reduces to a linear function of the Christoffel scalar curvature when the torsion is set to zero.

Theorem V.2 also shows that a gravitational Lagrangian which is a strictly local scalar function of the metric, Cartan curvature, and torsion also satisfies the condition:

(iv) The gravitational Lagrangian is a strictly local scalar function of $\theta^{\alpha}{}_{a}$, $\partial_c\theta^{\alpha}{}_{a}$, $\Gamma^{\alpha}{}_{\beta c}$, and $\partial_d\Gamma^{\alpha}{}_{\beta c}$.

Condition (iv) implies (iii). However, (iii) does not imply (iv) because, for example, the Christoffel scalar curvature (i.e. Einstein's Lagrangian) satisfies (iii) but not (iv).

I wish to conjecture two converses to Theorem V.2. First, it may be that any gravitational Lagrangian satisfying (iv) must be a strictly local function of the metric, Cartan curvature, and torsion. Second, it may be that any scalar gravitational Lagrangian satisfying (iii) must be the sum of a multiple of the Christoffel scalar curvature and of a strictly local function of the metric, Cartan curvature, and torsion. I have not yet tried to prove either of these conjectures.

Theorem V.1 deals with a smaller class of gravitational Lagrangians, namely those which are strictly local scalar functions of the metric and Cartan curvature only (no explicit torsion). These satisfy two conditions:

(v) The Einstein equations (variation with respect to $\theta^{\alpha}{}_{a}$) involve no derivatives of $\theta^{\alpha}{}_{a}$ and no higher than first derivatives of $\Gamma^{\alpha}{}_{\beta c}$. The Cartan equations (variation with respect to $\Gamma^{\alpha}{}_{\beta c}$) involve no higher than first derivatives of $\theta^{\alpha}{}_{a}$ and second derivatives of $\Gamma^{\alpha}{}_{\beta c}$.

(vi) The gravitational Lagrangian is a strictly local scalar function of $\theta^{\alpha}{}_{a}$, $\Gamma^{\alpha}{}_{\beta c}$, and $\partial_d\Gamma^{\alpha}{}_{\beta c}$.

Condition (vi) implies (v). I conjecture that any scalar gravitational Lagrangian satisfying (v) also satisfies (vi) and that any gravitational Lagrangian satisfying (vi) must be a strictly local scalar function of the metric and Cartan curvature. I have not tried to prove these conjectures.

Theorems V.1 and V.2 also show that if one wishes to regard $g_{ab}$ and $Q^a{}_{bc}$ (or $\lambda^a{}_{bc}$) as the independent variables instead of $\theta^\alpha{}_a$ and $\Gamma^\alpha{}_{\beta a}$, then one obtains an equivalent set of field equations although they contain higher than second derivatives of $g_{ab}$ and $Q^a{}_{bc}$.

I conclude this subsection by stating and proving the two theorems I have been discussing.

*Theorem V.1:*

*Suppose the gravitational Lagrangian,* $L_G$ *, is a strictly local, scalar function of the metric and the Cartan curvature tensors. Then the Lagrangian density,*

$$\mathcal{L}_G = \sqrt{-\tilde{g}}\ L_G(g_{ab},\ \hat{R}^a{}_{bcd}), \tag{1}$$

*is a strictly local function of any of the following sets of variables:*

$$\mathcal{L}_G = \mathcal{L}^1_G(g_{ab},\ \partial_c g_{ab},\ \partial_d\partial_c g_{ab},\ Q^a{}_{bc},\ \partial_d Q^a{}_{bc}), \tag{2}$$

$$\mathcal{L}_G = \mathcal{L}^2_G(g_{ab},\ \partial_c g_{ab},\ \partial_d\partial_c g_{ab},\ \lambda^a{}_{bc},\ \partial_d \lambda^a{}_{bc}), \tag{3}$$

$$\mathcal{L}_G = \mathcal{L}^3_G(\theta^\alpha{}_a,\ \partial_c\theta^\alpha{}_a,\ \partial_d\partial_c\theta^\alpha{}_a,\ Q^\alpha{}_{\beta\gamma},\ \partial_d Q^\alpha{}_{\beta\gamma}), \tag{4}$$

$$\mathcal{L}_G = \mathcal{L}^4_G(\theta^\alpha{}_a,\ \partial_c\theta^\alpha{}_a,\ \partial_d\partial_c\theta^\alpha{}_a,\ \lambda^\alpha{}_{\beta\gamma},\ \partial_d \lambda^\alpha{}_{\beta\gamma}), \tag{5}$$

$$\mathcal{L}_G = \mathcal{L}^5_G(\theta^\alpha{}_a,\ \Gamma^\alpha{}_{\beta c},\ \partial_d\Gamma^\alpha{}_{\beta c}). \tag{6}$$

*Further, equivalent sets of vacuum field equations are obtained by varying* $g_{ab}$ *and* $Q^a{}_{bc}$ *in* $\mathcal{L}^1_G$, $g_{ab}$ *and* $\lambda^a{}_{bc}$ *in* $\mathcal{L}^2_G$, $\theta^\alpha{}_a$ *and* $Q^\alpha{}_{\beta\gamma}$ *in* $\mathcal{L}^3_G$, $\theta^\alpha{}_a$ *and* $\lambda^\alpha{}_{\beta\gamma}$ *in* $\mathcal{L}^4_G$, *or* $\theta^\alpha{}_a$ *and* $\Gamma^\alpha{}_{\beta c}$ *in* $\mathcal{L}^5_G$. *Furthermore, the* $\mathcal{L}^5_G$ *variations may be computed from*

$$\frac{\delta\mathcal{L}^5_G}{\delta\theta^\alpha{}_a} = \sqrt{-\tilde{g}}\left(2\ e_\mu{}^a\ \hat{R}^\rho{}_{\sigma\nu\alpha}\ \frac{\partial L_G}{\partial\hat{R}^\rho{}_{\sigma\mu\nu}} + e_\alpha{}^a\ L_G\right), \tag{7}$$

$$\frac{\delta\mathcal{L}^5_G}{\delta\Gamma^\alpha{}_{\beta c}} = 2\ \sqrt{-\tilde{g}}\ \ \nabla_d\left(\frac{\partial L_G}{\partial\hat{R}^\alpha{}_{\beta\mu\nu}}\ e_\mu{}^c\ e_\nu{}^d\right). \tag{8}$$

*Proof:*

In a coordinate basis, $\partial_a$ and $dx^a$, the Christoffel connection, defect tensor, Cartan connection, Cartan curvature, and metric determinant are

$$\{^a{}_{bc}\} = \frac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}), \tag{9}$$

$$\lambda^a{}_{bc} = \frac{1}{2} g^{ad}(g_{be}Q^e{}_{dc} + g_{ce}Q^e{}_{db} - g_{de}Q^e{}_{bc}), \tag{10}$$

$$\Gamma^a{}_{bc} = \{^a{}_{bc}\} + \lambda^a{}_{bc} , \tag{11}$$

$$\hat{R}^a{}_{bcd} = \partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc} , \tag{12}$$

$$\tilde{g} = \det g_{ab}. \tag{13}$$

An examination of equations (9) through (13) in conjunction with equation (1) shows that $\mathcal{L}_G$ is a strictly local function of the variables shown in equations (2) and (3).

Similarly, in the orthonormal bases, $e_\alpha = e_\alpha{}^a\partial_a$ and $\theta^\alpha = \theta^\alpha{}_a\, dx^a$, substituting

$$g_{ab} = \eta_{\alpha\beta}\, \theta^\alpha{}_a\, \theta^\beta{}_b , \tag{14}$$

$$Q^a{}_{bc} = e_\alpha{}^a\, \theta^\beta{}_b\, \theta^\gamma{}_c\, Q^\alpha{}_{\beta\gamma} , \tag{15}$$

$$\lambda^a{}_{bc} = e_\alpha{}^a\, \theta^\beta{}_b\, \theta^\gamma{}_c\, \lambda^\alpha{}_{\beta\gamma} , \tag{16}$$

into equations (2) and (3) yields equations (4) and (5).

Since $L_G$ is a scalar, it is independent of the choice of frame. Hence, it may be regarded as a strictly local function either of the coordinate components of the metric, $g_{ab}$, and the Cartan curvature, $\hat{R}^a{}_{bcd}$, or of their orthonormal components, $g_{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(s,-s,-s,-s)$, and $\hat{R}^\alpha{}_{\beta\gamma\delta}$. Since the $g_{\alpha\beta}$ are constants, $L_G$ is a strictly local function of $\hat{R}^\alpha{}_{\beta\gamma\delta}$ only. Hence,

$$\mathcal{L}_G = \sqrt{-\tilde{g}}\, L_G(\hat{R}^\alpha{}_{\beta\gamma\delta}). \tag{17}$$

In terms of the mixed components of the Cartan connection, $\Gamma^\alpha{}_{\beta c}$, the mixed components of the Cartan curvature are

$$\hat{R}^\alpha{}_{\beta cd} = \partial_c \Gamma^\alpha{}_{\beta d} - \partial_d \Gamma^\alpha{}_{\beta c} + \Gamma^\alpha{}_{\varepsilon c} \Gamma^\varepsilon{}_{\beta d} - \Gamma^\alpha{}_{\varepsilon d} \Gamma^\varepsilon{}_{\beta c}, \tag{18}$$

so that the orthonormal components of the Cartan curvature are

$$\hat{R}^\alpha{}_{\beta\gamma\delta} = e_\gamma{}^c\, e_\delta{}^d\, \hat{R}^\alpha{}_{\beta cd}\ . \tag{19}$$

The determinant of the coordinate components of the metric may also be written as

$$\tilde{g} = (\det g_{\alpha\beta})(\det \theta^\alpha{}_a)^2 = -(\det \theta^\alpha{}_a)^2. \tag{20}$$

An examination of equations (18), (19) and (20) in conjunction with equation (17) shows that $\mathcal{L}_G$ is a strictly local function of the variables shown in equation (6).

The vacuum field equations for $\mathcal{L}^1_G$ are

$$\frac{\delta \mathcal{L}^1_G}{\delta g_{ab}} = \frac{\partial \mathcal{L}^1_G}{\partial g_{ab}} - \partial_c \frac{\partial \mathcal{L}^1_G}{\partial \partial_c g_{ab}} + \partial_d \partial_c \frac{\partial \mathcal{L}^1_G}{\partial \partial_d \partial_c g_{ab}} = 0\ , \tag{21}$$

$$\frac{\delta \mathcal{L}^1_G}{\delta Q^a{}_{bc}} = \frac{\partial \mathcal{L}^1_G}{\partial Q^a{}_{bc}} - \partial_d \frac{\partial \mathcal{L}^1_G}{\partial \partial_d Q^a{}_{bc}} = 0; \tag{22}$$

for $\mathcal{L}_G^2$ are

$$\frac{\delta\mathcal{L}_G^2}{\delta g_{ab}} = \frac{\partial\mathcal{L}_G^2}{\partial g_{ab}} - \partial_c \frac{\partial\mathcal{L}_G^2}{\partial\partial_c g_{ab}} + \partial_d\partial_c \frac{\partial\mathcal{L}_G^2}{\partial\partial_d\partial_c g_{ab}} = 0 \ , \tag{23}$$

$$\frac{\delta\mathcal{L}_G^2}{\delta\lambda^a{}_{bc}} = \frac{\partial\mathcal{L}_G^2}{\partial\lambda^a{}_{bc}} - \partial_d \frac{\partial\mathcal{L}_G^2}{\partial\partial_d\lambda^a{}_{bc}} = 0 \ ; \tag{24}$$

for $\mathcal{L}_G^3$ are

$$\frac{\delta\mathcal{L}_G^3}{\delta\theta^\alpha{}_a} = \frac{\partial\mathcal{L}_G^3}{\partial\theta^\alpha{}_a} - \partial_c \frac{\partial\mathcal{L}_G^3}{\partial\partial_c\theta^\alpha{}_a} + \partial_d\partial_c \frac{\partial\mathcal{L}_G^3}{\partial\partial_d\partial_c\theta^\alpha{}_a} = 0, \tag{25}$$

$$\frac{\delta\mathcal{L}_G^3}{\delta Q^\alpha{}_{\beta\gamma}} = \frac{\partial\mathcal{L}_G^3}{\partial Q^\alpha{}_{\beta\gamma}} - \partial_d \frac{\partial\mathcal{L}_G^3}{\partial\partial_d Q^\alpha{}_{\beta\gamma}} = 0; \tag{26}$$

for $\mathcal{L}_G^4$ are

$$\frac{\delta\mathcal{L}_G^4}{\delta\theta^\alpha{}_a} = \frac{\partial\mathcal{L}_G^4}{\partial\theta^\alpha{}_a} - \partial_c \frac{\partial\mathcal{L}_G^4}{\partial\partial_c\theta^\alpha{}_a} + \partial_d\partial_c \frac{\partial\mathcal{L}_G^4}{\partial\partial_d\partial_c\theta^\alpha{}_a} = 0, \tag{27}$$

$$\frac{\delta\mathcal{L}_G^4}{\delta\lambda^\alpha{}_{\beta\gamma}} = \frac{\partial\mathcal{L}_G^4}{\partial\lambda^\alpha{}_{\beta\gamma}} - \partial_d \frac{\partial\mathcal{L}_G^4}{\partial\partial_d\lambda^\alpha{}_{\beta\gamma}} = 0 \ ; \tag{28}$$

and for $\mathcal{L}_G^5$ are

$$\frac{\delta\mathcal{L}_G^5}{\delta\theta^\alpha{}_a} = \frac{\partial\mathcal{L}_G^5}{\partial\theta^\alpha{}_a} = 0, \tag{29}$$

$$\frac{\delta\mathcal{L}_G^5}{\delta\Gamma^\alpha{}_{\beta c}} = \frac{\partial\mathcal{L}_G^5}{\partial\Gamma^\alpha{}_{\beta c}} - \partial_d \frac{\partial\mathcal{L}_G^5}{\partial\partial_d\Gamma^\alpha{}_{\beta c}} = 0. \tag{30}$$

The equations obtained by varying g or θ are called Einstein equations, while those obtained by varying Q, λ or Γ are called Cartan equations. The various sets of equations may be related using the chain rule as follows.

First, the variations of $\mathcal{L}_G^3$ and $\mathcal{L}_G^4$ may be related using

$$Q^{\kappa}{}_{\mu\nu} = \lambda^{\kappa}{}_{\nu\mu} - \lambda^{\kappa}{}_{\mu\nu} = S^{\kappa}{}_{\mu\nu\rho}{}^{\sigma\tau}\,\lambda^{\rho}{}_{\sigma\tau} \quad , \tag{31}$$

$$\partial_q Q^{\kappa}{}_{\mu\nu} = S^{\kappa}{}_{\mu\nu\rho}{}^{\sigma\tau}\,\partial_q \lambda^{\rho}{}_{\sigma\tau} \quad , \tag{32}$$

where by definition,

$$S^{\kappa}{}_{\mu\nu\rho}{}^{\sigma\tau} = \frac{1}{2}(\delta^{\kappa}_{\rho}\,\delta^{\sigma}_{\nu}\,\delta^{\tau}_{\mu} - \delta^{\kappa}_{\rho}\,\delta^{\sigma}_{\mu}\,\delta^{\tau}_{\nu} - g^{\kappa\sigma}\,g_{\nu\rho}\,\delta^{\tau}_{\mu} + g^{\kappa\sigma}\,g_{\mu\rho}\,\delta^{\tau}_{\nu}). \tag{33}$$

By chain rule

$$\frac{\delta\mathcal{L}_G^4}{\delta\lambda^{\alpha}{}_{\beta\gamma}} = \frac{\partial\mathcal{L}_G^3}{\partial Q^{\kappa}{}_{\mu\nu}}\,\frac{\partial Q^{\kappa}{}_{\mu\nu}}{\partial\lambda^{\alpha}{}_{\beta\gamma}} - \partial_d\left(\frac{\partial\mathcal{L}_G^3}{\partial\partial_q Q^{\kappa}{}_{\mu\nu}}\,\frac{\partial\partial_q Q^{\kappa}{}_{\mu\nu}}{\partial\partial_d\lambda^{\alpha}{}_{\beta\gamma}}\right) \tag{34}$$

$$= \frac{\delta\mathcal{L}_G^3}{\delta Q^{\kappa}{}_{\mu\nu}}\,S^{\kappa}{}_{\mu\nu\alpha}{}^{\beta\gamma} \quad . \tag{35}$$

This may be inverted to give

$$\frac{\delta\mathcal{L}_G^3}{\delta Q^{\alpha}{}_{\beta\gamma}} = \frac{\delta\mathcal{L}_G^4}{\delta\lambda^{\rho}{}_{\sigma\tau}}\,T^{\rho}{}_{\sigma\tau\alpha}{}^{\beta\gamma} \quad , \tag{36}$$

where by definition,

$$T^{\rho}{}_{\sigma\tau\alpha}{}^{\beta\gamma} = \frac{1}{4}\,(g_{\sigma\alpha}\,g^{\rho\beta}\,\delta^{\gamma}_{\tau} + g_{\tau\alpha}\,g^{\rho\beta}\,\delta^{\gamma}_{\sigma} - \delta^{\rho}_{\alpha}\,\delta^{\beta}_{\sigma}\,\delta^{\gamma}_{\tau} - g_{\sigma\alpha}\,g^{\rho\gamma}\,\delta^{\beta}_{\tau} - g_{\tau\alpha}\,g^{\rho\gamma}\,\delta^{\beta}_{\sigma} + \delta^{\rho}_{\alpha}\,\delta^{\gamma}_{\sigma}\,\delta^{\beta}_{\tau}). \tag{37}$$

Since the formula (31) for $Q^{\kappa}{}_{\mu\nu}$ in terms of $\lambda^{\rho}{}_{\sigma\tau}$ is independent of $\theta^{\alpha}{}_{a}$,

$$\frac{\delta \mathcal{L}_G^4}{\delta \theta^{\alpha}{}_a} = \frac{\delta \mathcal{L}_G^3}{\delta \theta^{\alpha}{}_a} \quad . \tag{38}$$

Thus the $\mathcal{L}_G^3$ equations are satisfied iff the $\mathcal{L}_G^4$ equations are satisfied.

Second, the variations of $\mathcal{L}_G^1$ and $\mathcal{L}_G^3$ may be related using

$$g_{rs} = g_{\rho\sigma}\, \theta^{\rho}{}_r\, \theta^{\sigma}{}_s \, , \tag{39}$$

$$\partial_p g_{rs} = g_{\rho\sigma}[(\partial_p \theta^{\rho}{}_r)\, \theta^{\sigma}{}_s + \theta^{\rho}{}_r(\partial_p \theta^{\sigma}{}_s)], \tag{40}$$

$$\partial_q \partial_p g_{rs} = g_{\rho\sigma}[(\partial_q \partial_p \theta^{\rho}{}_r)\theta^{\sigma}{}_s + (\partial_p \theta^{\rho}{}_r)\partial_q \theta^{\sigma}{}_s + (\partial_q \theta^{\rho}{}_r)\partial_p \theta^{\sigma}{}_s + \theta^{\rho}{}_r \partial_q \partial_p \theta^{\sigma}{}_s], \tag{41}$$

$$Q^{r}{}_{st} = e_{\rho}{}^{r}\theta^{\sigma}{}_s \theta^{\tau}{}_t\, Q^{\rho}{}_{\sigma\tau}, \tag{42}$$

$$\partial_p Q^{r}{}_{st} = e_{\rho}{}^{r}\theta^{\sigma}{}_s \theta^{\tau}{}_t\, \partial_p Q^{\rho}{}_{\sigma\tau} + \frac{\partial(e_{\rho}{}^{r}\theta^{\sigma}{}_s\theta^{\tau}{}_t)}{\partial\theta^{\eta}{}_h}\, (\partial_p \theta^{\eta}{}_h)\, Q^{\rho}{}_{\sigma\tau} \quad . \tag{43}$$

By chain rule,

$$\begin{aligned}
\frac{\delta \mathcal{L}_G^3}{\delta \theta^{\alpha}{}_a} = {} & \frac{\partial \mathcal{L}_G^1}{\partial g_{rs}} \frac{\partial g_{rs}}{\partial \theta^{\alpha}{}_a} + \frac{\partial \mathcal{L}_G^1}{\partial \partial_p g_{rs}} \frac{\partial \partial_p g_{rs}}{\partial \theta^{\alpha}{}_a} + \frac{\partial \mathcal{L}_G^1}{\partial \partial_q \partial_p g_{rs}} \frac{\partial \partial_q \partial_p g_{rs}}{\partial \theta^{\alpha}{}_a} \\
& + \frac{\partial \mathcal{L}_G^1}{\partial Q^{r}{}_{st}} \frac{\partial Q^{r}{}_{st}}{\partial \theta^{\alpha}{}_a} + \frac{\partial \mathcal{L}_G^1}{\partial \partial_p Q^{r}{}_{st}} \frac{\partial \partial_p Q^{r}{}_{st}}{\partial \theta^{\alpha}{}_a} \\
& - \partial_c \left( \frac{\partial \mathcal{L}_G^1}{\partial \partial_p g_{rs}} \frac{\partial \partial_p g_{rs}}{\partial \partial_c \theta^{\alpha}{}_a} + \frac{\partial \mathcal{L}_G^1}{\partial \partial_q \partial_p g_{rs}} \frac{\partial \partial_q \partial_p g_{rs}}{\partial \partial_c \theta^{\alpha}{}_a} + \frac{\partial \mathcal{L}_G^1}{\partial \partial_p Q^{r}{}_{st}} \frac{\partial \partial_p Q^{r}{}_{st}}{\partial \partial_c \theta^{\alpha}{}_a} \right) \\
& + \partial_d \partial_c \left( \frac{\partial \mathcal{L}_G^1}{\partial \partial_q \partial_p g_{rs}} \frac{\partial \partial_q \partial_p g_{rs}}{\partial \partial_d \partial_c \theta^{\alpha}{}_a} \right)
\end{aligned} \tag{44}$$

$$= 2\, \frac{\delta \mathcal{L}_G^1}{\delta g_{ab}}\, g_{\alpha\beta}\, \theta^{\beta}{}_b + \frac{\delta \mathcal{L}_G^1}{\delta Q^{r}{}_{st}} (2\, \delta^{a}_{s}\, e_{\alpha}{}^{p}\, Q^{r}{}_{pt} - e_{\alpha}{}^{r}\, Q^{a}{}_{st}\, ), \tag{45}$$

$$\frac{\delta \mathcal{L}_G^3}{\delta Q^\alpha{}_{\beta\gamma}} = \frac{\partial \mathcal{L}_G^1}{\partial Q^r{}_{st}} \frac{\partial Q^r{}_{st}}{\partial Q^\alpha{}_{\beta\gamma}} + \frac{\partial \mathcal{L}_G^1}{\partial \partial_p Q^r{}_{st}} \frac{\partial \partial_p Q^r{}_{st}}{\partial Q^\alpha{}_{\beta\gamma}}$$

$$- \partial_d \left( \frac{\partial \mathcal{L}_G^1}{\partial \partial_p Q^r{}_{st}} \frac{\partial \partial_p Q^r{}_{st}}{\partial \partial_d Q^\alpha{}_{\beta\gamma}} \right) \tag{46}$$

$$= \frac{\delta \mathcal{L}_G^1}{\delta Q^r{}_{st}} e_\alpha{}^r \theta^\beta{}_s \theta^\gamma{}_t . \tag{47}$$

These may be inverted to give

$$\frac{\delta \mathcal{L}_G^1}{\delta g_{ab}} = \frac{1}{2} \frac{\delta \mathcal{L}_G^3}{\delta \theta^\alpha{}_a} \theta^\alpha{}_c g^{cb} + \frac{\delta \mathcal{L}_G^3}{\delta Q^\alpha{}_{\beta\gamma}} (\tfrac{1}{2} \theta^\alpha{}_r g^{rb} e_\delta{}^a Q^\delta{}_{\beta\gamma} - \theta^\delta{}_r g^{rb} e_\beta{}^a Q^\alpha{}_{\delta\gamma}), \tag{48}$$

$$\frac{\delta \mathcal{L}_G^1}{\delta Q^a{}_{bc}} = \frac{\delta \mathcal{L}_G^3}{\delta Q^\alpha{}_{\beta\gamma}} \theta^\alpha{}_a e_\beta{}^b e_\gamma{}^c . \tag{49}$$

Thus the $\mathcal{L}_G^1$ equations are satisfied iff the $\mathcal{L}_G^3$ equations are satisfied.

Note that the antisymmetric part of equation (48) is an identity which expresses the fact that $\mathcal{L}_G^3$ is independent of the choice of orthonormal frame. It may be derived by a technique similar to that used in Section III.5c to derive the conservation laws via Noether's theorem.

Replacing $\mathcal{L}_G^1$ by $\mathcal{L}_G^2$, $\mathcal{L}_G^3$ by $\mathcal{L}_G^4$ and Q by $\lambda$ in equations (39) through (49) shows that the $\mathcal{L}_G^2$ equations are satisfied iff the $\mathcal{L}_G^4$ equations are satisfied.

It remains to prove that the $\mathcal{L}_G^4$ equations are equivalent to the $\mathcal{L}_G^5$ equations and to derive the $\mathcal{L}_G^5$ equations. It is first necessary to express $\Gamma^\alpha{}_{\beta c}$ in terms of $\theta^\alpha{}_a$ and $\lambda^\alpha{}_{\beta\gamma}$. The commutator functions are defined by

$$[e_\beta, e_\gamma] = c^\alpha{}_{\beta\gamma} e_\alpha , \tag{50}$$

so that

$$c^{\alpha}{}_{\beta\gamma} = (\delta^{\mu}_{\beta}\,\delta^{\nu}_{\gamma} - \delta^{\nu}_{\beta}\,\delta^{\mu}_{\gamma})e_{\mu}{}^{m}\,e_{\nu}{}^{n}\,\partial_{n}\theta^{\alpha}{}_{m}\,. \tag{51}$$

The orthonormal components of the Christoffel and Cartan connections are then

$$\{^{\rho}{}_{\sigma\tau}\} = \frac{1}{2}\,g^{\rho\eta}(g_{\sigma\kappa}c^{\kappa}{}_{\eta\tau} + g_{\tau\kappa}c^{\kappa}{}_{\eta\sigma} - g_{\eta\kappa}\,c^{\kappa}{}_{\sigma\tau}) \tag{52}$$

$$= 2\,T^{\rho}{}_{\sigma\tau\kappa}{}^{\mu\nu}\,e_{\mu}{}^{m}\,e_{\nu}{}^{n}\,\partial_{n}\theta^{\kappa}{}_{m}\,, \tag{53}$$

$$\Gamma^{\rho}{}_{\sigma\tau} = \{^{\rho}{}_{\sigma\tau}\} + \lambda^{\rho}{}_{\sigma\tau} \tag{54}$$

$$= 2\,T^{\rho}{}_{\sigma\tau\kappa}{}^{\mu\nu}\,e_{\mu}{}^{m}\,e_{\nu}{}^{n}\,\partial_{n}\theta^{\kappa}{}_{m} + \lambda^{\rho}{}_{\sigma\tau}\,, \tag{55}$$

where $T^{\rho}{}_{\sigma\tau\kappa}{}^{\mu\nu}$ is given in equation (37). Hence the mixed components of the Cartan connection and its derivative are,

$$\Gamma^{\rho}{}_{\sigma t} = \theta^{\tau}{}_{t}\,\Gamma^{\rho}{}_{\sigma\tau} \tag{56}$$

$$= 2\,T^{\rho}{}_{\sigma\tau\kappa}{}^{\mu\nu}\,\theta^{\tau}{}_{t}\,e_{\mu}{}^{m}\,e_{\nu}{}^{n}\,\partial_{n}\theta^{\kappa}{}_{m} + \theta^{\tau}{}_{t}\,\lambda^{\rho}{}_{\sigma\tau}, \tag{57}$$

$$\partial_{p}\Gamma^{\rho}{}_{\sigma t} = 2\,T^{\rho}{}_{\sigma\tau\kappa}{}^{\mu\nu}\,\theta^{\tau}{}_{t}\,e_{\mu}{}^{m}\,e_{\nu}{}^{n}\,\partial_{p}\partial_{n}\theta^{\kappa}{}_{m}$$

$$+ 2\,T^{\rho}{}_{\sigma\tau\kappa}{}^{\mu\nu}\,\frac{\partial(\theta^{\tau}{}_{t}\,e_{\mu}{}^{m}\,e_{\nu}{}^{n})}{\partial\theta^{\eta}{}_{h}}(\partial_{p}\theta^{\eta}{}_{h})\partial_{n}\theta^{\kappa}{}_{m}$$

$$+ \lambda^{\rho}{}_{\sigma\tau}\,\partial_{p}\theta^{\tau}{}_{t} + \theta^{\tau}{}_{t}\,\partial_{p}\lambda^{\rho}{}_{\sigma\tau}\,. \tag{59}$$

Thus the $\mathcal{L}_G^4$ equations are satisfied iff the $\mathcal{L}_G^5$ equations are satisfied.

Finally, using chain rule and equations (29), (30), (17), (20), (18) and (19),

$$\frac{\delta \mathcal{L}_G^5}{\delta \theta^\alpha{}_a} = \frac{\partial \sqrt{-\tilde{g}}}{\partial \theta^\alpha{}_a} L_G + \sqrt{-\tilde{g}} \frac{\partial L_G}{\partial \hat{R}^\rho{}_{\sigma\mu\nu}} \frac{\partial \hat{R}^\rho{}_{\sigma\mu\nu}}{\partial \theta^\alpha{}_a} \tag{66}$$

$$= \sqrt{-\tilde{g}} \left( 2\, e_\mu{}^a \hat{R}^\rho{}_{\sigma\nu\alpha} \frac{\partial L_G}{\partial \hat{R}^\rho{}_{\sigma\mu\nu}} + e_\alpha{}^a L_G \right), \tag{67}$$

$$\frac{\delta \mathcal{L}_G^5}{\delta \Gamma^\alpha{}_{\beta c}} = \sqrt{-\tilde{g}} \frac{\partial L_G}{\partial \hat{R}^\rho{}_{\sigma\mu\nu}} \frac{\partial \hat{R}^\rho{}_{\sigma\mu\nu}}{\partial \Gamma^\alpha{}_{\beta c}} - \partial_d \left( \sqrt{-\tilde{g}} \frac{\partial L_G}{\partial \hat{R}^\rho{}_{\sigma\mu\nu}} \frac{\partial \hat{R}^\rho{}_{\sigma\mu\nu}}{\partial \partial_d \Gamma^\alpha{}_{\beta c}} \right) \tag{68}$$

$$= 2 \sqrt{-\tilde{g}}\, \nabla_d \left( \frac{\partial L_G}{\partial \hat{R}^\alpha{}_{\beta\mu\nu}} e_\mu{}^c e_\nu{}^d \right). \tag{69}$$

Q.E.D.

*Theorem V.2:*

*Suppose the gravitational Lagrangian, $L_G$, is a strictly local, scalar function of the metric, the Cartan curvature and the torsion tensors. Then the Lagrangian density,*

$$\mathcal{L}_G = \sqrt{-\tilde{g}}\, L_G(g_{ab},\ \hat{R}^a{}_{bcd},\ Q^a{}_{bc}), \tag{70}$$

*is strictly local function of any of the following sets of variables:*

$$\mathcal{L}_G = \mathcal{L}^1_G(g_{ab},\ \partial_c g_{ab},\ \partial_d\partial_c g_{ab},\ Q^a{}_{bc},\ \partial_d Q^a{}_{bc}), \tag{71}$$

$$\mathcal{L}_G = \mathcal{L}^2_G(g_{ab},\ \partial_c g_{ab},\ \partial_d\partial_c g_{ab},\ \lambda^a{}_{bc},\ \partial_d \lambda^a{}_{bc}); \tag{72}$$

$$\mathcal{L}_G = \mathcal{L}^3_G(\theta^\alpha{}_a,\ \partial_c\theta^\alpha{}_a,\ \partial_d\partial_c\theta^\alpha{}_a,\ Q^\alpha{}_{\beta\gamma},\ \partial_d Q^\alpha{}_{\beta\gamma}), \tag{73}$$

$$\mathcal{L}_G = \mathcal{L}^4_G(\theta^\alpha{}_a,\ \partial_c\theta^\alpha{}_a,\ \partial_d\partial_c\theta^\alpha{}_a,\ \lambda^\alpha{}_{\beta\gamma},\ \partial_d \lambda^\alpha{}_{\beta\gamma}), \tag{74}$$

$$\mathcal{L}_G = \mathcal{L}^5_G(\theta^\alpha{}_a,\ \partial_c\theta^\alpha{}_a,\ \Gamma^\alpha{}_{\beta c},\ \partial_d \Gamma^\alpha{}_{\beta c}). \tag{75}$$

*Further, equivalent sets of vacuum field equations are obtained by varying* $g_{ab}$ *and* $Q^a{}_{bc}$ *in* $\mathcal{L}^1_G$, $g_{ab}$ *and* $\lambda^a{}_{bc}$ *in* $\mathcal{L}^2_G$, $\theta^\alpha{}_a$ *and* $Q^\alpha{}_{\beta\gamma}$ *in* $\mathcal{L}^3_G$, $\theta^\alpha{}_a$ *and* $\lambda^\alpha{}_{\beta\gamma}$ *in* $\mathcal{L}^4_G$, *or* $\theta^\alpha{}_a$ *and* $\Gamma^\alpha{}_{\beta c}$ *in* $\mathcal{L}^5_G$. *Furthermore, the* $\mathcal{L}^5_G$ *variations may be computed from*

$$\frac{\delta\mathcal{L}^5_G}{\delta\theta^\alpha{}_a} = \sqrt{-\tilde{g}}\left[2\,\nabla_c\left(\frac{\partial L_G}{\partial Q^\alpha{}_{\mu\nu}}\, e_\mu{}^a\, e_\nu{}^c\right) + 2\, e_\mu{}^a\, Q^\rho{}_{\nu\alpha}\frac{\partial L_G}{\partial Q^\rho{}_{\mu\nu}} + 2\, e_\mu{}^a\, \hat{R}^\rho{}_{\sigma\nu\alpha}\frac{\partial L_G}{\partial \hat{R}^\rho{}_{\sigma\mu\nu}} + e_\alpha{}^a\, L_G\right], \tag{76}$$

$$\frac{\delta\mathcal{L}^5_G}{\delta\Gamma^\alpha{}_{\beta c}} = \sqrt{-\tilde{g}}\left[2\,\nabla_d\left(\frac{\partial L_G}{\partial \hat{R}^\alpha{}_{\beta\mu\nu}}\, e_\mu{}^c\, e_\nu{}^d\right) + e_\mu{}^c(\delta^\rho_\alpha\delta^\beta_\nu - g^{\rho\beta}g_{\nu\alpha})\frac{\partial L_G}{\partial Q^\rho{}_{\mu\nu}}\right]. \tag{77}$$

Proof:

The proof is essentially the same as that given for Theorem V.1. So I will only list the changes:

(i) Equation (17) is replaced by

$$\mathcal{L}_G = \sqrt{-\tilde{g}}\, L_G(\hat{R}^{\alpha}{}_{\beta\gamma\delta},\, Q^{\alpha}{}_{\beta\gamma}). \tag{78}$$

(ii) To see that $\mathcal{L}_G$ is a strictly local function of the variables shown in (75), the torsion must be expressed as

$$Q^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\gamma\beta} - \Gamma^{\alpha}{}_{\beta\gamma} - c^{\alpha}{}_{\beta\gamma} \tag{79}$$

$$= e_{\beta}{}^{b}\Gamma^{\alpha}{}_{\gamma b} - e_{\gamma}{}^{c}\Gamma^{\alpha}{}_{\beta c} - (\delta^{\mu}_{\beta}\delta^{\nu}_{\gamma} - \delta^{\nu}_{\beta}\delta^{\mu}_{\gamma})e_{\mu}{}^{m}\, e_{\nu}{}^{n}\partial_{n}\theta^{\alpha}{}_{m}. \tag{80}$$

(iii) Equation (29) is replaced by

$$\frac{\delta\mathcal{L}^5_G}{\delta\theta^{\alpha}{}_{a}} = \frac{\partial\mathcal{L}^5_G}{\partial\theta^{\alpha}{}_{a}} - \partial_c \frac{\partial\mathcal{L}^5_G}{\partial\partial_c\theta^{\alpha}{}_{a}} = 0. \tag{81}$$

(iv) The term,

$$-\,\partial_c \frac{\partial\mathcal{L}^5_G}{\partial\partial_c\theta^{\alpha}{}_{a}} \quad , \tag{82}$$

must be added to equation (60) although equation (61) remains unchanged.

(v) Finally, equations (66) through (69) are replaced by

$$\frac{\delta\mathcal{L}^5_G}{\delta\theta^{\alpha}{}_{a}} = \frac{\partial\sqrt{-\tilde{g}}}{\partial\theta^{\alpha}{}_{a}} L_G + \sqrt{-\tilde{g}}\,\frac{\partial L_G}{\partial\hat{R}^{\rho}{}_{\sigma\mu\nu}}\,\frac{\partial\hat{R}^{\rho}{}_{\sigma\mu\nu}}{\partial\theta^{\alpha}{}_{a}}$$

$$+ \sqrt{-\tilde{g}}\,\frac{\partial L_G}{\partial Q^{\rho}{}_{\mu\nu}}\,\frac{\partial Q^{\rho}{}_{\mu\nu}}{\partial\theta^{\alpha}{}_{a}} - \partial_c\left(\sqrt{-\tilde{g}}\,\frac{\partial L_G}{\partial Q^{\rho}{}_{\mu\nu}}\,\frac{\partial Q^{\rho}{}_{\mu\nu}}{\partial\partial_c\theta^{\alpha}{}_{a}}\right) \tag{83}$$

$$= \sqrt{-\tilde{g}} \left[ 2 \nabla_c \left( \frac{\partial L_G}{\partial Q^\alpha{}_{\mu\nu}} e_\mu{}^a e_\nu{}^c \right) + 2 e_\mu{}^a Q^\rho{}_{\nu\alpha} \frac{\partial L_G}{\partial Q^\rho{}_{\mu\nu}} \right.$$

$$\left. + 2 e_\mu{}^a \hat{R}^\rho{}_{\sigma\nu\alpha} \frac{\partial L_G}{\partial \hat{R}^\rho{}_{\sigma\mu\nu}} + e_\alpha{}^a L_G \right] , \tag{84}$$

$$\frac{\delta \mathcal{L}^5_G}{\delta \Gamma^\alpha{}_{\beta c}} = \sqrt{-\tilde{g}} \frac{\partial L_G}{\partial \hat{R}^\rho{}_{\sigma\mu\nu}} \frac{\partial \hat{R}^\rho{}_{\sigma\mu\nu}}{\partial \Gamma^\alpha{}_{\beta c}} + \sqrt{-\tilde{g}} \frac{\partial L_G}{\partial Q^\rho{}_{\mu\nu}} \frac{\partial Q^\rho{}_{\mu\nu}}{\partial \Gamma^\alpha{}_{\beta c}}$$

$$- \partial_d \left( \sqrt{-\tilde{g}} \frac{\partial L_G}{\partial \hat{R}^\rho{}_{\sigma\mu\nu}} \frac{\partial \hat{R}^\rho{}_{\sigma\mu\nu}}{\partial \partial_d \Gamma^\alpha{}_{\beta c}} \right) \tag{85}$$

$$= \sqrt{-\tilde{g}} \left[ 2 \nabla_d \left( \frac{\partial L_G}{\partial \hat{R}^\alpha{}_{\beta\mu\nu}} e_\mu{}^c e_\nu{}^d \right) + e_\mu{}^c (\delta^\rho_\alpha \delta^\beta_\nu - g^{\rho\beta} g_{\nu\alpha}) \frac{\partial L_G}{\partial Q^\rho{}_{\mu\nu}} \right] . \tag{86}$$

Q.E.D.

c. Automatic Noether Conservation Laws, But No Additional Automatic Conservation Laws

To say that a theory of gravity has automatic Noether conservation laws means that the gravitational field equations together with certain differential identities on the gravitational variables imply that the matter variables satisfy the Noether conservation laws even if the matter field equations, energy-momentum tensor and spin tensor were not derived from a matter Lagrangian.

When there is a matter Lagrangian, $L_M$, then the canonical energy-momentum tensor, $t_\alpha{}^a$, the canonical spin tensor, $S^\beta{}_\alpha{}^a$, and the Euler-Lagrange tensor, $L_{(X)}$, are defined by

$$-s\sqrt{-g}\,t_\alpha{}^a = \frac{\delta\mathcal{L}_M}{\delta\theta^\alpha{}_a}, \tag{87}$$

$$-s\sqrt{-g}\,\frac{1}{2}S^\beta{}_\alpha{}^a = \frac{\delta\mathcal{L}_M}{\delta\Gamma^\alpha{}_{\beta a}}, \tag{88}$$

$$\sqrt{-g}\,L_{(X)} = \frac{\delta\mathcal{L}_M}{\delta\psi^{(X)}}. \tag{89}$$

Here $\mathcal{L}_M = \sqrt{-g}\,L_M$ is the matter Lagrangian density. In a metric-Cartan connection theory, if $L_M$ is a scalar and the matter field equations,

$$L_{(X)} = 0, \tag{90}$$

are satisfied, then $t_\alpha{}^a$ and $S^\beta{}_\alpha{}^a$ satisfy the Noether conservation laws:

$$\nabla_a t_\gamma{}^a = \frac{1}{2}S^\beta{}_\alpha{}^\delta\,\hat{R}^\alpha{}_{\beta\gamma\delta} + t_\alpha{}^\delta\,Q^\alpha{}_{\gamma\delta}, \tag{91}$$

$$\nabla_a S_{\beta\alpha}{}^a = 2\,t_{[\beta\alpha]}. \tag{92}$$

(See Section III.5.c)

In this subsection I show that the Noether equations (91) and (92) may be rederived without using equations (87) through (90). Instead, I assume that $t_\alpha{}^a$ and $S^\beta{}_\alpha{}^a$ are defined phenomenologically and that the gravitational field equations of the metric-Cartan connection theory may be written as

$$E_\alpha{}^a = t_\alpha{}^a \quad , \tag{93}$$

$$C^\beta{}_\alpha{}^a = S^\beta{}_\alpha{}^a \quad , \tag{94}$$

where $E_\alpha{}^a$ and $C^\beta{}_\alpha{}^a$ depend only on the gravitational variables. I refer to equations (93) as the Einstein equations and to equations (94) as the Cartan equations. I call $E_\alpha{}^a$ the <u>Einstein field tensor</u> and call $C^\beta{}_\alpha{}^a$ the <u>Cartan field tensor</u>. Suppose $E_\alpha{}^a$ and $C^\beta{}_\alpha{}^a$ satisfy identities of the form,

$$\nabla_a E_\gamma{}^a = \frac{1}{2} C^\beta{}_\alpha{}^\delta \, \hat{R}^\alpha{}_{\beta\gamma\delta} + E_\alpha{}^\delta \, Q^\alpha{}_{\gamma\delta} \quad , \tag{95}$$

$$\nabla_a C_{\beta\alpha}{}^a = 2 \, E_{[\beta\alpha]} \quad . \tag{96}$$

Then (91) and (92) would follow from (93) and (94), and the theory would have automatic Noether conservation laws.

When are (95) and (96) satisfied? I will deal with the case when $E_\alpha{}^a$ and $C^\beta{}_\alpha{}^a$ are defined by

$$s \sqrt{-\tilde{g}} \, E_\alpha{}^a = \frac{\delta \mathcal{L}_G}{\delta \theta^\alpha{}_a} \quad , \tag{97}$$

$$s \sqrt{-\tilde{g}} \, \frac{1}{2} C^\beta{}_\alpha{}^a = \frac{\delta \mathcal{L}_G}{\delta \Gamma^\alpha{}_{\beta a}} \quad , \tag{98}$$

where $\mathcal{L}_G = \sqrt{-\tilde{g}} \, L_G$ is a gravitational Lagrangian density.

One special case has $L_G = -s \frac{\hbar c}{16\pi L^2} \tilde{R}$ where $\tilde{R}$ is the Christoffel scalar curvature. Then $E_\alpha{}^a = \frac{\hbar c}{8\pi L^2} \tilde{G}_\alpha{}^a$, and $C^\beta{}_\alpha{}^a = 0$, where $\tilde{G}_{\alpha\beta}$ is the Christoffel Einstein curvature tensor. In this case equations (95) and (96) reduce to the identities

$$\nabla_a \tilde{G}^{ba} = 0 \quad , \quad \tilde{G}_{[ba]} = 0 \, . \tag{99}$$

The second case I consider is treated in the following theorem.

*Theorem V.3:*

*If the gravitational Lagrangian,*

$$L_G = L_G(\theta^\alpha{}_a, \partial_b\theta^\alpha{}_a, \Gamma^\alpha{}_{\beta a}, \partial_b\Gamma^\alpha{}_{\beta a}), \tag{100}$$

*is a scalar function, then equations* (95) *and* (96) *are satisfied and the theory has automatic Noether conservation laws.*

The proof is almost identical to the proof of Noether's theorem (see Section III.5c) and will not be given here.

Since definitions (97) and (98) are linear in $L_G$ and equations (95) and (96) are linear in $E_\alpha{}^a$ and $C^\beta{}_\alpha{}^a$, Theorems V.2 and V.3 imply that the Lagrangian,

$$L_G = -s \frac{\hbar c}{16\pi L^2} c_2 \tilde{R} + \bar{L}(g_{ab}, \hat{R}^a{}_{bcd}, Q^a{}_{bc}), \tag{101}$$

provides automatic Noether conservation laws as well as second order field equations.

I would also like to impose the condition that the gravitational field equations do *not* imply any conservation laws other than the Noether conservation laws. Unfortunately, I am unable to formulate this condition precisely. The best I can do is to give an example. Consider the gravitational Lagrangian,

$$L_G = - s \frac{\hbar c}{16\pi L^2} \tilde{R} - \frac{\hbar c}{16\pi\alpha_G} \hat{R}^{\alpha}{}_{\beta\gamma\delta} \hat{R}^{\beta}{}_{\alpha}{}^{\gamma\delta} . \tag{102}$$

This differs from the Lagrangian considered in Chapter VI only in that it uses $\tilde{R}$ instead of $\hat{R}$. For Lagrangian (102),

$$E_{\mu\nu} = \frac{\hbar c}{8\pi L^2} \tilde{G}_{\mu\nu} + s \frac{\hbar c}{4\pi\alpha_G} (\hat{R}^{\alpha}{}_{\beta\gamma\mu} \hat{R}^{\beta}{}_{\alpha}{}^{\gamma}{}_{\nu} - \frac{1}{4} g_{\mu\nu} \hat{R}^{\alpha}{}_{\beta\gamma\delta} \hat{R}^{\beta}{}_{\alpha}{}^{\gamma\delta} ), \tag{103}$$

$$C^{\alpha}{}_{\beta}{}^{a} = - s \frac{\hbar c}{2\pi\alpha_G} \nabla_b \hat{R}^{\alpha}{}_{\beta}{}^{ab} , \tag{104}$$

satisfy (95) and (96) and hence there are automatic Noether conservation laws. But they also satisfy the identities,

$$E_{[\mu\nu]} = 0 , \tag{105}$$

$$\nabla_a C^{\alpha}{}_{\beta}{}^{a} = - s \frac{\hbar c}{2\pi\alpha_G} \nabla_a \nabla_b \hat{R}^{\alpha}{}_{\beta}{}^{ab} = 0. \tag{106}$$

Using equations (93) and (94) this says

$$\nabla_a S_{\alpha\beta}{}^{a} = t_{[\alpha\beta]} = 0 ; \tag{107}$$

i.e. there is separate conservation of orbital and spin angular momentum. This is undesirable. Hence, I rule out the theory of gravity based on Lagrangian (102).

d. Quadratic Polynomial Lagrangians

The gravitational Lagrangian (101) describes a very large class of metric-Cartan connection theories of gravity. One can reduce the number by requiring the gravitational Lagrangian to be a polynomial in $\hat{R}^{\alpha}{}_{\beta\gamma\delta}$ and $Q^{\alpha}{}_{\beta\gamma}$. In this subsection, I find and discuss the most general gravitational Lagrangian which is a quadratic polynomial in $\hat{R}^{\alpha}{}_{\beta\gamma\delta}$ and $Q^{\alpha}{}_{\beta\gamma}$ plus a multiple of the Christoffel scalar curvature, $\tilde{R}$.

Thus, I assume the gravitational Lagrangian has the form

$$L_G = -\frac{\hbar c}{8\pi L^2}\Lambda - s\frac{\hbar c}{16\pi L^2}(c_1\hat{R} + c_2\tilde{R}) + \frac{\hbar c}{16\pi}A^{\alpha\beta\gamma\delta\kappa\lambda\mu\nu}\hat{R}_{\alpha\beta\gamma\delta}\hat{R}_{\kappa\lambda\mu\nu} - s\frac{\hbar c}{16\pi L^2}B^{\alpha\gamma\delta\kappa\mu\nu}Q_{\alpha\gamma\delta}Q_{\kappa\mu\nu} \quad . \tag{108}$$

Here (i) $\Lambda$ is a cosmological constant of dimensions ( length )$^{-2}$, (ii) $L = (\hbar G/c^3)^{\frac{1}{2}}$ is the Planck length, (iii) $c_1$ and $c_2$ are dimensionless coupling constants, and (iv) $A^{\alpha\beta\gamma\delta\kappa\lambda\mu\nu}$ and $B^{\alpha\gamma\delta\kappa\mu\nu}$ are dimensionless tensors constructed solely out of the metric. The problem is to determine $A^{\alpha\beta\gamma\delta\kappa\lambda\mu\nu}$ and $B^{\alpha\gamma\delta\kappa\mu\nu}$.

Since the Cartan connection is metric-compatible, the Cartan curvature, $\hat{R}_{\alpha\beta\gamma\delta}$, is antisymmetric in $(\alpha\beta)$ and separately in $(\gamma\delta)$. Hence, it is sufficient to assume that $A^{\alpha\beta\gamma\delta\kappa\lambda\mu\nu}$ is separately antisymmetric in each of the pairs $(\alpha\beta)$, $(\gamma\delta)$, $(\kappa\lambda)$ and $(\mu\nu)$ and symmetric under the quadruple interchange of $(\alpha\beta\gamma\delta)$ with $(\kappa\lambda\mu\nu)$. Similarly, $Q_{\alpha\gamma\delta}$ is antisymmetric in $(\gamma\delta)$. So it is sufficient to assume $B^{\alpha\gamma\delta\kappa\mu\nu}$ is separately antisymmetric in $(\gamma\delta)$ and in $(\mu\nu)$ and symmetric under the triple interchange of $(\alpha\gamma\delta)$ with $(\kappa\mu\nu)$.

## TABLE V.1: COMPUTATION OF CARTAN CURVATURE SQUARED LAGRANGIANS

| add $\alpha\beta$ | add $\gamma\delta$ | add $\kappa\lambda$ | add $\mu\nu$ | $A_1^{\alpha\beta\gamma\delta\kappa\lambda\mu\nu}$ | $A_1^{\alpha\beta\gamma\delta\kappa\lambda\mu\nu}\hat{R}_{\alpha\beta\gamma\delta}\hat{R}_{\kappa\lambda\mu\nu}$ |
|---|---|---|---|---|---|
| $g^{\alpha-}g^{\beta-}g^{--}g^{--}$ | $g^{\alpha\gamma}g^{\beta\delta}g^{--}g^{--}$ | $\longrightarrow g^{\alpha\gamma}g^{\beta\delta}g^{\kappa-}g^{\lambda-}$ | $\longrightarrow g^{\alpha\gamma}g^{\beta\delta}g^{\kappa\mu}g^{\lambda\nu}$ | $= A_1^{\alpha\beta\gamma\delta\kappa\lambda\mu\nu}$ | $\hat{R}\ \hat{R}$ |
| | | $g^{\alpha\gamma}g^{\beta\kappa}g^{\delta\lambda}g^{--}$ | $\longrightarrow 0$ | | |
| | $g^{\alpha\gamma}g^{\beta-}g^{\delta-}g^{--}$ | $\longrightarrow g^{\alpha\gamma}g^{\beta\kappa}g^{\delta-}g^{\lambda-}$ | $\longrightarrow g^{\alpha\gamma}g^{\beta\kappa}g^{\delta\mu}g^{\lambda\nu}$ | $= A_2^{\alpha\beta\gamma\delta\kappa\lambda\mu\nu}$ | $\hat{R}_{\beta\delta}\ \hat{R}^{\beta\delta}$ |
| | | $g^{\alpha\gamma}g^{\beta-}g^{\delta\kappa}g^{\lambda-}$ | $\longrightarrow g^{\alpha\gamma}g^{\beta\mu}g^{\delta\kappa}g^{\lambda\nu}$ | $= A_4^{\alpha\beta\gamma\delta\kappa\lambda\mu\nu}$ | $\hat{R}_{\beta\delta}\ \hat{R}^{\delta\beta}$ |
| | | $g^{\alpha\kappa}g^{\beta\lambda}g^{\gamma-}g^{\delta-}$ | $\longrightarrow g^{\alpha\kappa}g^{\beta\lambda}g^{\gamma\mu}g^{\delta\nu}$ | $= A_3^{\alpha\beta\gamma\delta\kappa\lambda\mu\nu}$ | $\hat{R}_{\alpha\beta\gamma\delta}\ \hat{R}^{\alpha\beta\gamma\delta}$ |
| | $g^{\alpha-}g^{\beta-}g^{\gamma-}g^{\delta-}$ | $\longrightarrow g^{\alpha\kappa}g^{\beta-}g^{\gamma\lambda}g^{\delta-}$ | $\longrightarrow g^{\alpha\kappa}g^{\beta\mu}g^{\gamma\lambda}g^{\delta\nu}$ | $= A_6^{\alpha\beta\gamma\delta\kappa\lambda\mu\nu}$ | $\hat{R}_{\alpha\beta\gamma\delta}\ \hat{R}^{\alpha\gamma\beta\delta}$ |
| | | $g^{\alpha-}g^{\beta-}g^{\gamma\kappa}g^{\delta\lambda}$ | $\longrightarrow g^{\alpha\mu}g^{\beta\nu}g^{\gamma\kappa}g^{\delta\lambda}$ | $= A_5^{\alpha\beta\gamma\delta\kappa\lambda\mu\nu}$ | $\hat{R}_{\alpha\beta\gamma\delta}\ \hat{R}^{\gamma\delta\alpha\beta}$ |

⌐¬ denotes antisymmetrization. $\hat{R}_{\beta\delta} = \hat{R}^{\gamma}{}_{\beta\gamma\delta}$. $\hat{R} = \hat{R}^{\delta}{}_{\delta}$.

## TABLE V.2: COMPUTATION OF TORSION SQUARED LAGRANGIANS

| add $\alpha$ | add $\gamma\delta$ | add $\kappa$ | add $\mu\nu$ | $B_1^{\alpha\gamma\delta\kappa\mu\nu}$ | $B_1^{\alpha\gamma\delta\kappa\mu\nu} Q_{\alpha\gamma\delta} Q_{\kappa\mu\nu}$ |
|---|---|---|---|---|---|
| $g^{\alpha-}\ g^{--}\ g^{--}$ | $g^{\alpha\gamma}\ g^{\delta-}\ g^{--}$ | $g^{\alpha\gamma}\ g^{\delta\kappa}\ g^{--} \longrightarrow 0$ | | | |
| | | $g^{\alpha\gamma}\ g^{\delta-}\ g^{\kappa-} \longrightarrow$ | $g^{\alpha\gamma}\ g^{\delta\mu}\ g^{\kappa\nu}$ | $= -B_1^{\alpha\gamma\delta\kappa\mu\nu}$ | $Q_\delta\ Q^\delta$ |
| | $g^{\alpha-}\ g^{\gamma-}\ g^{\delta-}$ | $g^{\alpha\kappa}\ g^{\gamma-}\ g^{\delta-} \longrightarrow$ | $g^{\alpha\kappa}\ g^{\gamma\mu}\ g^{\delta\nu}$ | $= B_2^{\alpha\gamma\delta\kappa\mu\nu}$ | $Q^\alpha{}_{\gamma\delta}\ Q_\alpha{}^{\gamma\delta}$ |
| | | $g^{\alpha-}\ g^{\gamma\kappa}\ g^{\delta-} \longrightarrow$ | $g^{\alpha\mu}\ g^{\gamma\kappa}\ g^{\delta\nu}$ | $= B_3^{\alpha\gamma\delta\kappa\mu\nu}$ | $Q^\alpha{}_{\gamma\delta}\ Q^\gamma{}_\alpha{}^\delta$ |

⌐¬ denotes antisymmetrization. $Q_\delta = Q^\gamma{}_{\gamma\delta}$.

Since $A^{\alpha\beta\gamma\delta\kappa\lambda\mu\nu}$ (resp. $B^{\alpha\gamma\delta\kappa\mu\nu}$) must be constructed solely out of the metric, it must be a homogeneous polynomial of degree four (resp. three) in the metric, $g^{\rho\sigma}$, where in each term the indices $\alpha\beta\gamma\delta\kappa\lambda\mu\nu$ (resp. $\alpha\gamma\delta\kappa\mu\nu$) are permuted among the g's. There are relations among the coefficients due to the symmetries of $A^{\alpha\beta\gamma\delta\kappa\lambda\mu\nu}$ (resp. $B^{\alpha\gamma\delta\kappa\mu\nu}$). Thus $A^{\alpha\beta\gamma\delta\kappa\lambda\mu\nu}$ (resp. $B^{\alpha\gamma\delta\kappa\mu\nu}$) is a linear combination of certain independent polynomials. These independent polynomials are derived and listed in Table V.1 (resp. Table V.2 ). The method of computation is to write down four (resp. three) g's and to successively add indices in all possible independent positions, antisymmetrizing when necessary.

Thus, the most general form of the gravitational Lagrangian (108) is

$$\begin{aligned} L_G = &- \frac{\hbar c}{8\pi L^2} \Lambda - s \frac{\hbar c}{16\pi L^2} (c_1 \hat{R} + c_2 \tilde{R}) \\ &+ \frac{\hbar c}{16\pi} (a_1 \hat{R} \hat{R} + a_2 \hat{R}_{\beta\delta} \hat{R}^{\beta\delta} + a_3 \hat{R}_{\alpha\beta\gamma\delta} \hat{R}^{\alpha\beta\gamma\delta} \\ &\qquad + a_4 \hat{R}_{\beta\delta} \hat{R}^{\delta\beta} + a_5 \hat{R}_{\alpha\beta\gamma\delta} \hat{R}^{\gamma\delta\alpha\beta} + a_6 \hat{R}_{\alpha\beta\gamma\delta} \hat{R}^{\alpha\gamma\beta\delta}) \\ &- s \frac{\hbar c}{16\pi L^2} (b_1 Q^{\alpha}{}_{\alpha\delta} Q^{\gamma}{}_{\gamma}{}^{\delta} + b_2 Q^{\alpha}{}_{\gamma\delta} Q_{\alpha}{}^{\gamma\delta} + b_3 Q^{\alpha}{}_{\gamma\delta} Q^{\gamma}{}_{\alpha}{}^{\delta}). \end{aligned} \tag{109}$$

There is still one relation among the quadratic Cartan curvature scalars. Consider the scalar,

$$\begin{aligned} P &= \varepsilon_{\alpha\beta\kappa\lambda} \, \varepsilon^{\gamma\delta\mu\nu} \, \hat{R}^{\alpha\beta}{}_{\gamma\delta} \, \hat{R}^{\kappa\lambda}{}_{\mu\nu} \\ &= -4(\hat{R} \hat{R} - 4 \hat{R}_{\alpha\beta} \hat{R}^{\beta\alpha} + \hat{R}^{\alpha\beta}{}_{\gamma\delta} \hat{R}^{\gamma\delta}{}_{\alpha\beta}). \end{aligned} \tag{110}$$

On a 4-dimensional manifold, the integral,

$$\int P \sqrt{-\tilde{g}} \, d^4x \, , \tag{111}$$

is a topological invariant related to the Euler characteristic. As a topological invariant, this integral is invariant under continuous changes in the metric or connection. Hence the addition of any multiple of P to the Lagrangian cannot change the field equations. (The usual argument that P is a total divergence is only valid when spacetime is contractible; i.e. topologically $R^4$.) Comparison of (110) with (109) shows that one of the parameters, $a_1$, $a_4$ or $a_5$ is arbitrary.

I now proceed to discuss the class of theories based on the gravitational Lagrangian (109). By investigating the Newtonian limit for these theories one should be able to determine one parameter and put limits on the remaining parameters. Since the Newtonian gravitational constant, $G = L^2 c^3/\hbar$, only appears in the scalar curvature and quadratic torsion terms, the Newtonian limit should fix some combination of the $b_i$'s and the $c_i$'s. By analogy with the Einstein and ECSK theories, I would naively expect that $c_1 + c_2 = 1$. However, the theory being investigated by Hehl, Ne'eman, Nitsch and Von der Heyde has $c_1 = c_2 = b_3 = 0$, $b_2 = -1$, and $b_1 = 2$, and yet still has a long range Newtonian limit.

The field equations for Lagrangian (109) may be obtained by varying $\tilde{R}$ explicitly and using equations (76) and (77) of Theorem V.2 for the remaining terms.

The Einstein equation is

$$
\begin{aligned}
E_\alpha{}^a = &- s\frac{\hbar c}{8\pi L^2}\Lambda e_\alpha{}^a + \frac{\hbar c}{8\pi L^2}(c_1\hat{G}^a{}_\alpha + c_2\tilde{G}^a{}_\alpha) \\
&- s\frac{\hbar c}{4\pi}a_1(\hat{R}\hat{R}^a{}_\alpha - \frac{1}{4}e_\alpha{}^a\hat{R}\hat{R}) \\
&+ s\frac{\hbar c}{8\pi}a_2(\hat{R}^{\mu\nu}\hat{R}^a{}_{\mu\nu\alpha} - \hat{R}^{\mu a}\hat{R}_{\mu\alpha} + \frac{1}{2}e_\alpha{}^a\hat{R}^{\mu\nu}\hat{R}_{\mu\nu}) \\
&- s\frac{\hbar c}{4\pi}a_3(\hat{R}^{\kappa\lambda\mu a}\hat{R}_{\kappa\lambda\mu\alpha} - \frac{1}{4}e_\alpha{}^a\hat{R}^{\kappa\lambda\mu\nu}\hat{R}_{\kappa\lambda\mu\nu}) \\
&+ s\frac{\hbar c}{8\pi}a_4(\hat{R}^{\nu\mu}\hat{R}^a{}_{\mu\nu\alpha} - \hat{R}^{a\mu}\hat{R}_{\mu\alpha} + \frac{1}{2}e_\alpha{}^a\hat{R}^{\nu\mu}\hat{R}_{\mu\nu}) \\
&- s\frac{\hbar c}{4\pi}a_5(\hat{R}^{\mu a\kappa\lambda}\hat{R}_{\kappa\lambda\mu\alpha} - \frac{1}{4}e_\alpha{}^a\hat{R}^{\mu\nu\kappa\lambda}\hat{R}_{\kappa\lambda\mu\nu}) \\
&+ s\frac{\hbar c}{8\pi}a_6(\hat{R}^{\kappa a\lambda\mu}\hat{R}_{\kappa\lambda\mu\alpha} - \hat{R}^{\kappa\mu\lambda a}\hat{R}_{\kappa\lambda\mu\alpha} + \frac{1}{2}e_\alpha{}^a\hat{R}^{\kappa\mu\lambda\nu}\hat{R}_{\kappa\lambda\mu\nu}) \\
&+ \frac{\hbar c}{8\pi L^2}b_1(e_\alpha{}^c e_\beta{}^a\nabla_c Q^\gamma{}_\gamma{}^\beta - e_\alpha{}^a\nabla_c Q^\gamma{}_\gamma{}^c - \frac{1}{2}e_\alpha{}^a Q^\gamma{}_{\gamma\delta}Q^\beta{}_\beta{}^\delta) \\
&- \frac{\hbar c}{4\pi L^2}b_2(\nabla_c Q_\alpha{}^{ac} - Q^{\kappa\mu a}Q_{\kappa\mu\alpha} + \frac{1}{4}e_\alpha{}^a Q^{\kappa\mu\nu}Q_{\kappa\mu\nu}) \\
&- \frac{\hbar c}{4\pi L^2}b_3(\nabla_c Q^{[a}{}_\alpha{}^{c]} + Q^{[a}{}_\kappa{}^{c]}Q^\kappa{}_{c\alpha} + \frac{1}{2}e_\alpha{}^a Q^{\mu\kappa\nu}Q_{\kappa\mu\nu}) \\
&= t_\alpha{}^a \quad . \qquad (112)
\end{aligned}
$$

The Cartan equation is

$$
\begin{aligned}
C^{\beta}{}_{\alpha}{}^{c} = &- \frac{\hbar c}{4\pi L^2}\, c_1\, g^{\beta\gamma}\, \nabla_d (e_\alpha{}^{[c}\, e_\gamma{}^{d]}) \\
&+ s\, \frac{\hbar c}{2\pi}\, a_1\, g^{\beta\gamma}\, \nabla_d (\hat{R}\, e_\alpha{}^{[c}\, e_\gamma{}^{d]}) \\
&+ s\, \frac{\hbar c}{4\pi}\, a_2\, g^{\beta\gamma}\, \nabla_d (\hat{R}_\alpha{}^{[c}\, e_\gamma{}^{d]} - \hat{R}_\gamma{}^{[c}\, e_\alpha{}^{d]}) \\
&+ s\, \frac{\hbar c}{2\pi}\, a_3\, \nabla_d \hat{R}_\alpha{}^{\beta cd} \\
&+ s\, \frac{\hbar c}{4\pi}\, a_4\, g^{\beta\gamma}\, \nabla_d (\hat{R}^{[c}{}_\alpha\, e_\gamma{}^{d]} - \hat{R}^{[c}{}_\gamma\, e_\alpha{}^{d]}) \\
&+ s\, \frac{\hbar c}{2\pi}\, a_5\, \nabla_d \hat{R}^{cd}{}_\alpha{}^{\beta} \\
&+ s\, \frac{\hbar c}{4\pi}\, a_6\, g^{\beta\gamma}\, \nabla_d (\hat{R}_\alpha{}^{[c}{}_\gamma{}^{d]} - \hat{R}_\gamma{}^{[c}{}_\alpha{}^{d]}) \\
&- \frac{\hbar c}{4\pi L^2}\, b_1\, g^{\beta\delta}\, Q^{\gamma}{}_{\gamma[\alpha}\, e_{\delta]}{}^{c} \\
&- \frac{\hbar c}{2\pi L^2}\, b_2\, g^{\beta\delta}\, Q_{[\alpha}{}^{c}{}_{\delta]} \\
&- \frac{\hbar c}{4\pi L^2}\, b_3\, g^{\beta\delta} (Q^{c}{}_{\alpha\delta} - Q_{[\alpha\delta]}{}^{c}) \\
= &\; S^{\beta}{}_{\alpha}{}^{c} \quad . \qquad (113)
\end{aligned}
$$

In deriving and studying these equations it is useful to recall that

$$\nabla_c e_\alpha{}^a = -\lambda^a{}_{\alpha c} = -\frac{1}{2} g^{ab}(Q_{\alpha bc} + Q_{cb\alpha} - Q_{b\alpha c}), \tag{114}$$

$$Q^a{}_{bc} = \lambda^a{}_{cb} - \lambda^a{}_{bc} = \theta^\alpha{}_b \nabla_c e_\alpha{}^a - \theta^\alpha{}_c \nabla_b e_\alpha{}^a. \tag{115}$$

Notice that the field equations are linear in the second derivatives of the frame, $e_\alpha{}^a$, and the connection, $\Gamma^\alpha{}_{\beta a}$. Further, if $c_2 = b_1 = b_2 = b_3 = 0$ then there are no second derivatives of $e_\alpha{}^a$, while if $a_i = 0$, $i = 1 \ldots 6$, then there are no second derivatives of $\Gamma^\alpha{}_{\beta a}$.

From Theorem V.3, $E_\alpha{}^a$ and $C^\beta{}_\alpha{}^a$ satisfy equations (95) and (96) and there are automatic Noether conservation laws. In fact since $E_\alpha{}^a$ and $C^\beta{}_\alpha{}^a$ satisfy (95) and (96) for arbitrary values of $\Lambda$, $a_i$, $b_i$, and $c_i$, each term in $E_\alpha{}^a$ together with the corresponding term in $C^\beta{}_\alpha{}^a$ must satisfy (95) and (96) separately. Upon computing $E_{\alpha\beta}$, notice that the coefficients of $\Lambda$, $c_2$ and $a_3$ are symmetric in $\alpha$ and $\beta$. Hence, in order to avoid separate conservation of spin and orbital angular momentum, at least one of the remaining constants, $a_i$, $b_i$ or $c_i$ must be non-zero.

### e. Yang-Mills Analogy

The Lagrangian (109) still represents a large class of theories. It is desirable to investigate all of those theories, but one might use intuition to pick out a theory to study first. Since the Yang-Mills gauge theories are quantizable, one is led to ask which gravitational Lagrangian is most analogous to the Yang-Mills Lagrangian. I considered this question in Section II.5 and concluded that the Yang-Mills gravitational Lagrangian for a metric-Cartan connection theory is

$$L_G = - s \frac{\hbar c}{16\pi L^2} b_2 Q^{\alpha}{}_{ab} Q_{\alpha}{}^{ab} - \frac{\hbar c}{16\pi \alpha_G} \hat{R}^{\alpha}{}_{\beta ab} \hat{R}^{\beta}{}_{\alpha}{}^{ab} . \tag{116}$$

It is generally agreed that the curvature squared term is the appropriate Yang-Mills Lagrangian for the homogeneous part of the gravitational gauge group (O(3,1,R) or SL(2,C) or etc.). There is no such agreement that the torsion squared term is the appropriate Yang-Mills Lagrangian for the inhomogeneous part (the translation group or the diffeomorphism group or the coordinate transformation group).

In fact, the curvature squared term by itself is the gravitational Lagrangian of Yang's theory of gravity. Fairchild has shown that this theory does not have a satisfactory Newtonian limit. This is not surprising since the Newtonian gravitational constant, $G = L^2 c^3/\hbar$, does not appear in the Lagrangian. In order words, there is no length scale, $L$, in the problem. A length scale should come from the Lagrangian for the translation group.

I originally tried to use $\tilde{R}$ instead of $Q^{\alpha}{}_{ab} Q_{\alpha}{}^{ab}$ as the Yang-Mills Lagrangian for the translation group. (The justification is given in Section II.5.) However, as shown in Section V.3c, that Lagrangian leads to separate conservation of spin and orbital angular momentum.

I next tried to use $\hat{R}$ instead of $Q^{\alpha}{}_{ab}\, Q_{\alpha}{}^{ab}$. This is the Lagrangian considered by Mansouri and Chang [1976] and Fairchild [1977]. My results on this theory appear in Chapter VI.

I now consider (116) as the Lagrangian most analogous to the Yang-Mills Lagrangian but I have not begun to investigate that theory.

Recently, Hehl, Ne'eman, Nitsch and von der Heyde [1978] have argued that the Lagrangian,

$$L_G = s\,\frac{\hbar c}{16\pi L^2}\left(Q^{\alpha}{}_{\gamma\delta}\, Q_{\alpha}{}^{\gamma\delta} - 2\, Q^{\alpha}{}_{\alpha\delta}\, Q^{\gamma}{}_{\gamma}{}^{\delta}\right) - \frac{\hbar c}{16\pi\alpha_G}\,\hat{R}^{\alpha}{}_{\beta ab}\,\hat{R}^{\beta}{}_{\alpha}{}^{ab}, \qquad (117)$$

is the most analogous to the Yang-Mills Lagrangian. They show that in the weak field limit this theory has a long range Newtonian potential and a short range confining potential.

f. Unique Gravitational Ground State

It is desirable that a theory of gravity have a locally unique vacuum solution which is spatially homogeneous, isotropic and parity-invariant. This solution would be regarded as the ground state of the gravitational field. By "locally unique" I mean that the solution is unique up to identifications. Furthermore, it is desirable that this solution be Minkowski space so that the ground state would have no gravitational field. (For the Einstein theory with a cosmological constant, the ground state is de Sitter space, which has tidal forces.)

These requirements will be satisfied if the theory of gravity has a Birkhoff theorem which says that the only O(3)-spherically symmetric vacuum solution is Schwarzschild. This follows because any spatially homogeneous, isotropic and parity-invariant solution would have to be O(3)-spherically symmetric about every point; hence Schwarzschild about every point; hence Minkowski.

All of this is in contrast to the theory investigated by Horowitz and Wald, who showed that their field equations have homogeneous, isotropic vacuum solutions in addition to Minkowski.

g. Quantum Theory

At present there is no quantum theory of gravity. I consider the search for a quantum theory to be the most important reason to investigate theories of gravity other than Einstein's. The usual method in this search is to quantize a classical theory by perturbative techniques and then to check that the resulting quantum theory is unitary and renormalizable. Einstein's theory is unitary but non-renormalizable.

Recently, Stelle [1977 ] has demonstrated the renormalizability of the theory of gravity based on the Lagrangian,

$$L = - s \frac{\hbar c}{16\pi L^2} \tilde{R} + \alpha \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu} - \beta \tilde{R} \tilde{R} . \qquad (118)$$

In the context of metric theories, this is the most general scalar Lagrangian which is a quadratic polynomial in the Christoffel curvature (except for the addition of a cosmological constant). Unfortunately, Stelle's theory is non-unitary, essentially because the field equations contain third and fourth derivatives of the metric.

On the other hand, in the context of metric-Cartan connection theories, the Lagrangian (109) is the most general scalar Lagrangian which is a quadratic polynomial in the Cartan curvature, a quadratic polynomial in the torsion and a linear polynomial in the Christoffel curvature. Regarded as functions of the frame, $\theta^{\alpha}{}_{a}$, and the connection, $\Gamma^{\alpha}{}_{\beta c}$, the field equations (112) and (113) contain no higher than second derivatives of the frame and connection. Hence, the quantum theory based on Lagrangian (109) with $\theta^{\alpha}{}_{a}$ and $\Gamma^{\alpha}{}_{\beta c}$ as variables, may be unitary.

At the same time, regarded as a function of the metric, $g_{ab}$, and the defect, $\lambda^{a}{}_{bc}$, the Lagrangian (109) reduces to the Stelle Lagrangian for the metric, coupled to a Lagrangian for the defect. Thus the quantum theory based on Lagrangian (109) with $g_{ab}$ and $\lambda^{a}{}_{bc}$ as variables, has a chance of being renormalizable.

There is a danger here. It is well known that a change of variables in a classical theory can lead to inequivalent quantum theories. Theorem V.2 shows that Lagrangian (109) yields the same set of classical solutions whether the variables are chosen either as $\theta^{\alpha}{}_{a}$ and $\Gamma^{\alpha}{}_{\beta c}$ or as $g_{ab}$ and $\lambda^{a}{}_{bc}$. It is far from obvious that these two sets of variables lead to equivalent quantum theories. However, recall that for the Yang-Mills theory, unitarity and renormalizability were proven in different gauges and then proven to be gauge invariant properties of the quantum theory. Perhaps in the metric-Cartan connection theories it may be possible to prove unitarity using $\theta^{\alpha}{}_{a}$ and $\Gamma^{\alpha}{}_{\beta c}$, to prove renormalizability using $g_{ab}$ and $\lambda^{a}{}_{bc}$, and then to prove that these properties are invariant under this specific change of variables. I hope to return to this problem in the future.